MATH4060 Tutorial 5

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Problem 1 (Chap 6, Ex 2). *Prove that*

$$\prod_{n=1}^{\infty} \frac{n(n+a+b)}{(n+a)(n+b)} = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}$$

whenever a and b are positive. Using the product formula for $\sin \pi s$, give another proof that $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$.

We take the product representation of $\Gamma(s)$ as definition: for $a, b \neq -1, -2, \ldots$,

$$\begin{aligned}
\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)} &= \frac{a+b+1}{(a+1)(b+1)} e^{-\gamma} \prod_{n=1}^{\infty} \frac{n(n+a+b+1)}{(n+a+1)(n+b+1)} e^{1/n} \\
&= \frac{a+b+1}{(a+1)(b+1)} \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right) e^{-1/n} \prod_{n=1}^{\infty} \frac{n(n+a+b+1)}{(n+a+1)(n+b+1)} e^{1/n} \\
&= \frac{a+b+1}{(a+1)(b+1)} \prod_{n=1}^{\infty} \frac{(n+1)(n+a+b+1)}{(n+a+1)(n+b+1)} \\
&= \prod_{n=1}^{\infty} \frac{n(n+a+b)}{(n+a)(n+b)},
\end{aligned}$$

where we have used the identity $e^{-\gamma} = \prod_{n=1}^{\infty}(1+1/n)e^{-1/n}$ as in the proof of Theorem 1.7. Now, to prove the reflection formula, we first derive the functional equation. For $s \neq 0, -1, -2, \ldots$, take $a = s-1$ and $b = 1$ in the infinite product:

$$\frac{\Gamma(s)\Gamma(2)}{\Gamma(s+1)} = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{n(n+s)}{(n+s-1)(n+1)} = \lim_{N\to\infty} \frac{1}{s} \frac{N+s}{N+1} = \frac{1}{s}.$$

So we have $\Gamma(s+1) = s\Gamma(s)\Gamma(2) = s\Gamma(s)$ provided that $\Gamma(2) = 1$. This follows from the same identity, using $\Gamma(1) = 1$ and that $s\Gamma(s) \to 1$ as $s \to 0$. Then for the reflection formula, we take $a = s$ and $b = -s$:

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) = s^{-1}\Gamma(s+1)\Gamma(-s+1) &= s^{-1} \prod_{n=1}^{\infty} \frac{n^2}{(n+s)(n-s)} \\
&= s^{-1} \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right)^{-1} = \frac{\pi}{\sin \pi s}
\end{aligned}$$

as required.

Problem 2 (Chap 6, Ex 4). *Prove that if $f(z) = \frac{1}{(1-z)^{\alpha}}$ for $|z| < 1$, where $\alpha \in \mathbb{C}$ is fixed, then $f(z) = \sum_{n=0}^{\infty} a_n(\alpha) z^n$ with*

$$a_n(\alpha) \sim \frac{1}{\Gamma(\alpha)} n^{\alpha-1} \quad \textit{as } n \to \infty.$$

It suffices to show that $\lim_n a_n(\alpha)/n^{\alpha-1} = 1/\Gamma(\alpha)$. Without loss of generality, $\alpha \neq 0, -1, -2, \ldots$. By writing $f(z) = e^{-\alpha \log(1-z)}$ and noting that $f^{(n)}(z) = \alpha(\alpha+1)\cdots(\alpha+n-1)/(1-z)^{\alpha+n}$, we have

$$a_n(\alpha) = \frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}.$$

So

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n(\alpha)}{n^{\alpha-1}} &= \frac{1}{\alpha-1} \lim_{n\to\infty} \frac{(\alpha-1)((\alpha-1)+1)\cdots((\alpha-1)+n)}{n^{\alpha-1}n!} \\ &= \frac{1}{(\alpha-1)\Gamma(\alpha-1)} = \frac{1}{\Gamma(\alpha)},\end{aligned}$$

where we have used the representation of gamma function in Chap 6 Ex 1.

Problem 3 (Chap 6, Ex 15)**.** *Prove that for* $\mathrm{Re}(s) > 1$*,*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx.$$

Note that the integral is absolutely integrable: near zero because $e^x - 1 > x$ and $\mathrm{Re}(s) > 1$; and near infinity because of exponential decay. For $x > 0$, write $1/(e^x - 1) = e^{-x}/(1-e^{-x}) = \sum_{n=1}^\infty e^{-nx}$ so that

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}}{e^x-1}\, dx &= \int_0^\infty \sum_{n=1}^\infty e^{-nx} x^{s-1}\, dx = \sum_{n=1}^\infty \int_0^\infty e^{-nx}x^{s-1}\, dx \\ &= \sum_{n=1}^\infty \int_0^\infty n^{-s} e^{-x} x^{s-1}\, dx = \zeta(s)\Gamma(s).\end{aligned}$$

Here, the use of Fubini's theorem to interchange sum and integral is justified due to the absolute integrability and that $e^{-nx} > 0$ for all n; and we have used a change of variable $x \mapsto x/n$ for each term of the series.

Problem 4 (cf. Chap 6, Prob 1, 2)**.** *Show that* $|\zeta(1+it)| = O(\log|t|)$ *as* $|t| \to \infty$*.*

We first prove the following representation of $\zeta(s)$ on $\mathrm{Re}(s) > 0$: for every integer $N \geq 1$, we have

$$\zeta(s) = \sum_{n=1}^N n^{-s} - \frac{N^{1-s}}{1-s} - s\int_N^\infty \frac{\{x\}}{x^{s+1}}\, dx,$$

where $\{x\}$ denotes the fractional part of x. The idea is to first show that the identity holds for $\mathrm{Re}(s) > 1$, and then observe that the RHS defines a holomorphic function on $\mathrm{Re}(s) > 0$. For $\mathrm{Re}(s) > 1$, we write

$$\begin{aligned}\int_n^{n+1} \frac{\{x\}}{x^{s+1}}\, dx &= \int_n^{n+1} \frac{x}{x^{s+1}}\, dx - \int_n^{n+1} \frac{n}{x^{s+1}}\, dx \\ &= \frac{1}{1-s}\left[(n+1)^{1-s} - n^{1-s}\right] + \frac{n}{s}\left[(n+1)^{-s} - n^{-s}\right] \\ &= \left(\frac{1}{1-s} + \frac{1}{s}\right)\left[(n+1)^{1-s} - n^{1-s}\right] - \frac{1}{s}(n+1)^{-s} \\ &= \frac{1}{s(1-s)}\left[(n+1)^{1-s} - n^{1-s}\right] - \frac{1}{s}(n+1)^{-s}.\end{aligned}$$

Summing from $n = N$ gives

$$\int_N^\infty \frac{\{x\}}{x^{s+1}}\, dx = -\frac{N^{1-s}}{s(1-s)} - \frac{1}{s}\sum_{n=N+1}^\infty n^{-s}$$

because $\mathrm{Re}(s) > 1$ (so that both integral and telescoping series converge). Then

$$\zeta(s) = \sum_{n=1}^{N} n^{-s} + \sum_{n=N+1}^{\infty} n^{-s} = \sum_{n=1}^{N} n^{-s} - \frac{N^{1-s}}{1-s} - s\int_N^\infty \frac{\{x\}}{x^{s+1}}\,dx.$$

Using an argument similar to Proposition 2.5 and Corollary 2.6, observe that the integral[1] $\int_N^\infty \{x\}x^{-s-1}\,dx$ (viewed as an infinite sum as above) is uniformly convergent on any half-plane $\mathrm{Re}(s) \geq \epsilon$, $\epsilon > 0$. So the RHS gives an analytic continuation of $\zeta(s)$ to $\mathrm{Re}(s) > 0$. This proves the desired representation.

Next, for $s = \sigma + it$ with $\sigma > 0$ and $t \neq 0$, we have $|1-s| \geq |t|$ and

$$|\zeta(s)| \leq \sum_{n=1}^{N} n^{-\sigma} + \frac{N^{1-\sigma}}{|t|} + \frac{|s|N^{-\sigma}}{\sigma}.$$

Putting $\sigma = 1$ and $N = [|t|]$ with $|t| \geq 1$, we have $|s| \leq 2|t|$ and $N \leq |t| < N+1$. Therefore

$$|\zeta(s)| \leq \sum_{n=1}^{N} \frac{1}{n} + 1 + \frac{2(N+1)}{N} = O(\log|t|)$$

as $|t| \to \infty$, because $\sum_{n=1}^{N} n^{-1} = O(\log N)$, while the last two terms are bounded independent of $|t| \geq 1$.

Remarks: (a) We also have $|\zeta'(s)| = O(\log^2|t|)$ as $|t| \to \infty$, which can be proved using Cauchy integral formula[2], similar to Proposition 2.7, but using the above representation of $\zeta(s)$ instead. (b) For $s = 1 + it$, $t \neq 0$ fixed, we have

$$\sum_{n=1}^{N} \frac{1}{n^{1+it}} = \zeta(s) - \frac{N^{-it}}{it} + s\int_N^\infty \frac{\{x\}}{x^{s+1}}\,dx.$$

Observe that $|N^{-it}| = 1$ and the last term is bounded by $|s|/N$ which tends to 0 as $N \to \infty$. So partial sums of $\sum 1/n^{1+it}$ are bounded, but series does not converge because $N^{-it} = e^{-it\log N}$ does not. This shows that the series definition of $\zeta(s)$ cannot be extended to any point on $\mathrm{Re}(s) = 1$.

[1] i.e. write $\int_n^{n+1}\{x\}x^{-s-1} = \int_n^{n+1}(x^{-s} - n^{-s})\,dx - n\int_n^{n+1}(x^{-s-1} - n^{-s-1})\,dx$, then apply mean-value theorem as in the proposition/corollary.

[2] For an explicit proof, see Theorem 5.3 in `https://faculty.math.illinois.edu/~hildebr/ant/main.pdf`.